ISA Ph.D. Qualifying Exam for Fall 2012

Subject: Probability Theory

(10%) (1) Show that (a) $\sum_{k=0}^{n}(-1)^k \binom{n}{k} = 0$, (b) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

(10%) (2) Let an experiment be drawing five cards at random without replacement from a deck of 52 poker cards. The sample size is then $C(52,5) = 2,598,960$. Find the *size* of the following events.

(a) Four of a kind (four cards of equal face value and one card of a different value).

(b) Full house (one pair and one triple of cards with equal face value).

(c) Three of a kind (three equal face values plus two cards of different values).

(d) Two pairs (two pairs of equal face value plus one card of a different value).

(e) One pair (one pair of equal face value plus three cards of different values).

(10%) (3) Let the r.v. X have a Poisson distribution with the p.d.f. $f(x) = \lambda^x e^{-\lambda}/x!$, $x = 0, 1, 2, \ldots, \infty$, where $\lambda > 0$ is a known parameter.

(a) Find the mean, $E(X)$.

(b) Find the variance, Var(X).

(c) Find the mode of the probability density function f.

(20%) (4) Suppose that 2000 points are independently and randomly selected from the unit square $S = \{(x,y) : 0 \leq x, y \leq 1\}$. Let Y equal the number of points that fall in $R = \{(x,y) : |x+y| \leq 1 \text{ and } |x-y| \leq 1\}$.

(a) How is Y distributed?

(b) Give the mean and variance of Y.

(c) What is the expected value of $Y/500$?

(d) What is P($Y \leq 100$)?

(20%) (5) Suppose that the joint p.d.f. of two random variables X and Y is as follows:

$$f(x,y) = \begin{cases} k(x^2+y) & \text{for } 0 \leq \mathrm{y} \leq 1 - \mathrm{x}^2, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) Determine the value of the constant k.

(b) What is P($0 \leq X \leq 0.5$)?

(c) What is $\mathrm{P}(Y \leq X + 1)$?

(d) What is $\mathrm{P}(Y = X^2)$?

(10%) (6) Supoose that the joint p.d.f. of two random variables X and Y is as follows:

$$f(x,y) = \begin{cases} k(x+y^2) & \text{for } 0 \leq \mathrm{x} \leq 1 \text{ and } 0 \leq \mathrm{y} \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(a) Determine the conditional p.d.f. of X for any given value of Y.

(b) What is $\mathrm{P}(X < 0.5 | Y = 0.5)$?

(10%) (7) Suppose that three random variables X_1, X_2, and X_3 have a continuous joint distribution with the following joint p.d.f.:

$$f(x_1,x_2,x_3) = \begin{cases} ke^{-(x_1+2x_2+3x_3)} & \text{for } \mathrm{x_i} > 0 \ (\mathrm{i} = 1,2,3), \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

(a) Determine the marginal joint p.d.f. of X_1 and X_3.

(b) What is $\mathrm{P}(X_1 < 1 | X_2 = 2, X_3 = 1)$?

(10%) (8) Suppose that X_1 and X_2 are i.i.d. random variables and that each of them has a uniform distribution on the interval $(0,1)$. Find the p.d.f of $Y = X_1 + X_2$.