資應所 2012 年春季博士班資格考 機率論

1. (7%) (a) Describe Chebyshev's Theorem.
   (7%) (b) Prove the correctness of (a).

2. (10%) If X and Y are random variables with joint probability distribution $f(x, y)$, find the variance of $aX+bY$.

3. (10%) In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. If a product were chosen randomly and found to be defective, what is the probability that it was made by machine B_3?

4. (13%) Let X be the number of flips of a fair coin that are required to observe heads-tails on consecutive flips.
   (a) (4%) Find the probability mass function of X.
   (b) (4%) Find the value of $P(X \geq 5)$.
   (c) (5%) Determine the value of the mean of X.

5. (8%) Let X be binomially distributed with parameters n and p:
   $$f(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$
   Use the moment-generating function to show that its mean and variance are np and $np(1-p)$, respectively.

6. (12%) Let X and Y have the trinomial probability mass function with parameters n, p_1 and p_2:
   $$f(x, y) = \frac{n!}{x!\,y!\,(n-x-y)!} p_1^x p_2^y (1-p_1-p_2)^{n-x-y}$$
   where x and y are nonnegative integers such that $x + y \leq n$.
   (a) (4%) Find the conditional probability mass function of Y given $X = x$.
   (b) (4%) Find the conditional mean $E(Y|x)$ of Y given $X = x$.
   (c) (4%) Find the conditional probability mass function of X given $Y = y$.

7. (8%) Suppose that X has a normal distribution with mean μ. and variance σ^2.
   Express $E(X^3)$ in terms of μ. and σ^2.

8. (8%) Suppose that X has a normal distribution with zero mean and unit variance, and that the conditional distribution of Y given X is a normal distribution with mean $2X - 3$ and variance. Determine the mean and variance of the marginal distribution of Y.

9. (9%) Suppose that X has a continuous distribution for which the probability density function is as follows:

$$f(x) = \begin{cases} 0.5e^{-0.5\,x}, & \text{for } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the mean, the variance, and the median of this distribution.

10. (8%) Suppose that $X_1, \dots, X_n$ form a random sample from a distribution for which the probability density function is as follows:

$$f(x|\theta) = 0.5\, e^{-|x-\theta|}, \qquad -\infty < x < \infty.$$

Suppose that the value of θ is unknown $(-\infty < x < \infty)$. Find the maximum likelihood estimator of θ.